Solution to Problem Set 4

1. (a)

$$f'(x) = \frac{1}{2\sqrt{(x^2+5)(2x+3)}} \cdot [(2x)(2x+3) + (x^2+5)(2)]$$
$$= \frac{3x^2+3x+5}{\sqrt{(x^2+5)(2x+3)}}.$$

(b) Noting that $g(x) = \sqrt{ex^{e+1} + e^e x}$, we have

$$g'(x) = \frac{1}{2\sqrt{ex^{e+1}+e^e x}} \cdot [e(e+1)x^e + e^e] = \frac{e(e+1)x^e + e^e}{2\sqrt{x(ex^e+e^e)}}.$$

(c)

$$h'(x) = \left(\cos\sqrt{x}\right) \cdot \frac{1}{2\sqrt{x}} + \frac{1}{2\sqrt{\cos x}} \cdot (-\sin x) + \frac{1}{2\sqrt{\sin\sqrt{x}}} \cdot \left(\cos\sqrt{x}\right) \cdot \frac{1}{2\sqrt{x}}$$
$$= \frac{\cos\sqrt{x}}{2\sqrt{x}} - \frac{\sin x}{2\sqrt{\cos x}} + \frac{\cos\sqrt{x}}{4\sqrt{x\sin\sqrt{x}}}$$

(d) By product rule, we have

$$p'(x) = [(3^{\cos x}\ln 3)(-\sin x)]\left(e^{x-\sin(x^2)}\right) + (3^{\cos x})\left(e^{x-\sin(x^2)}(1-\cos(x^2)\cdot 2x)\right)$$
$$= \left(3^{\cos x}e^{x-\sin(x^2)}\right)[-(\ln 3)\sin x] + \left(3^{\cos x}e^{x-\sin(x^2)}\right)[1-2x\cos(x^2)]$$
$$= 3^{\cos x}e^{x-\sin(x^2)}[1-(\ln 3)\sin x - 2x\cos(x^2)].$$

Alternative solution: Noting that

$$p(x) = e^{(\ln 3)\cos x}e^{x-\sin(x^2)} = e^{(\ln 3)\cos x + x - \sin(x^2)},$$

we have by chain rule

$$p'(x) = e^{(\ln 3)\cos x + x - \sin(x^2)}[(\ln 3)(-\sin x) + 1 - \cos(x^2)\cdot 2x]$$
$$= 3^{\cos x}e^{x-\sin(x^2)}[1-(\ln 3)\sin x - 2x\cos(x^2)].$$

2. From the definition of e, we have

$$\lim_{h\to 0}\frac{e^h-1}{h} = 1.$$

With the change of variable $h = \ln(1+x)$, we have

$$1 = \lim_{x\to 0}\frac{(1+x)-1}{\ln(1+x)} = \lim_{x\to 0}\frac{x}{\ln(1+x)}.$$

Therefore

$$\lim_{x\to 0}\frac{\ln(1+x)}{x} = \frac{1}{1} = 1.$$

■

3. (a) Applying the change of variable $t = 2x$, we have

$$\lim_{x\to+\infty}\left(1+\frac{1}{2x}\right)^{x+1} = \lim_{x\to+\infty}\left(1+\frac{1}{t}\right)^{\frac{t}{2}+1} = \lim_{x\to+\infty}\left[\left(1+\frac{1}{t}\right)^{t}\right]^{\frac{1}{2}}\left(1+\frac{1}{t}\right)$$
$$= e^{\frac{1}{2}}(1+0) = \sqrt{e}.$$

(b) Applying the change of variable $t = \cot x$, we have both

$$\lim_{x\to0^+}(1+\tan x)^{\cot x} = \lim_{t\to+\infty}\left(1+\frac{1}{t}\right)^{t} = e \qquad \text{and} \qquad \lim_{x\to0^-}(1+\tan x)^{\cot x} = \lim_{t\to-\infty}\left(1+\frac{1}{t}\right)^{t} = e,$$

so

$$\lim_{x\to0}(1+\tan x)^{\cot x} = e.$$

(c) Since

$$\left(1+\frac{3}{x}+\frac{2}{x^2}\right)^{x} = \left(1+\frac{3x+2}{x^2}\right)^{x} = \left[\left(1+\frac{3x+2}{x^2}\right)^{\frac{x^2}{3x+2}}\right]^{\frac{3x+2}{x}} = \left[\left(1+\frac{3x+2}{x^2}\right)^{\frac{x^2}{3x+2}}\right]^{3+\frac{2}{x}},$$

we have

$$\lim_{x\to+\infty}\ln\left(1+\frac{3}{x}+\frac{2}{x^2}\right)^{x} = \lim_{x\to+\infty}\left(3+\frac{2}{x}\right)\ln\left(1+\frac{3x+2}{x^2}\right)^{\frac{x^2}{3x+2}} = (3+0)\ln e = 3$$

and since the exponential function is continuous, we have

$$\lim_{x\to+\infty}\left(1+\frac{3}{x}+\frac{2}{x^2}\right)^{x} = \lim_{x\to+\infty} e^{\ln\left(1+\frac{3}{x}+\frac{2}{x^2}\right)^{x}} = e^3.$$

(d) Consider the change of variable $t = \frac{x^2-3x-2}{x-1}$. Then $x = t+2+\frac{4}{x-1}$, so we have

$$\left(\frac{x^2-2x-3}{x^2-3x-2}\right)^{x} = \left(1+\frac{x-1}{x^2-3x-2}\right)^{x} = \left(1+\frac{1}{t}\right)^{t}\left(1+\frac{x-1}{x^2-3x-2}\right)^{2+\frac{4}{x-1}}.$$

Now since

$$\lim_{x\to+\infty}\ln\left(1+\frac{x-1}{x^2-3x-2}\right)^{2+\frac{4}{x-1}} = \lim_{x\to+\infty}\left(2+\frac{4}{x-1}\right)\ln\left(1+\frac{x-1}{x^2-3x-2}\right) = (2+0)\ln(1+0) = 0$$

and since the exponential function is continuous, we have

$$\lim_{x\to+\infty}\left(1+\frac{x-1}{x^2-3x-2}\right)^{2+\frac{4}{x-1}} = \lim_{x\to+\infty} e^{\ln\left(1+\frac{x-1}{x^2-3x-2}\right)^{2+\frac{4}{x-1}}} = e^0 = 1.$$

Therefore

$$\lim_{x\to+\infty}\left(\frac{x^2-2x-3}{x^2-3x-2}\right)^{x} = \lim_{t\to+\infty}\left(1+\frac{1}{t}\right)^{t}\lim_{x\to+\infty}\left(1+\frac{x-1}{x^2-3x-2}\right)^{2+\frac{4}{x-1}} = e\cdot1 = e.$$

4. (a) If the interest is compounded 4 times per year, then the total amount after t years is

$$A(t) = A_0\left(1+\frac{r}{4}\right)^{4t}.$$

If the interest is compounded 12 times per year, then the total amount after t years is

$$A(t) = A_0\left(1+\frac{r}{12}\right)^{12t}.$$

(b) In general, if the interest is compounded n times per year, then the total amount after t years is

$$A(t) = A_0\left(1+\frac{r}{n}\right)^{nt}.$$

Therefore, if the interest is compounded continuously, then the total amount after t years is

$$A(t) = \lim_{n\to+\infty} A_0\left(1+\frac{r}{n}\right)^{nt} = A_0 \lim_{n\to+\infty}\left[\left(1+\frac{r}{n}\right)^{\frac{n}{r}}\right]^{rt} = A_0\left[\lim_{n\to+\infty}\left(1+\frac{r}{n}\right)^{\frac{n}{r}}\right]^{rt} = A_0e^{rt}.$$

Specifically if $A_0 = 10000$ and $r = 0.06$, then the total amount $\$A$ after $t = 10$ years is given by

$$A = A_0e^{rt} = 10000e^{(0.06)(10)} \approx 18221.19.$$

5.

$$\frac{d}{dx}\sinh x = \frac{d}{dx}\frac{e^x - e^{-x}}{2} = \frac{1}{2}[e^x - (-e^{-x})] = \frac{e^x + e^{-x}}{2} = \cosh x$$

$$\frac{d}{dx}\cosh x = \frac{d}{dx}\frac{e^x + e^{-x}}{2} = \frac{1}{2}[e^x + (-e^{-x})] = \frac{e^x - e^{-x}}{2} = \sinh x$$

$$\frac{d}{dx}\tanh x = \frac{d}{dx}\frac{\sinh x}{\cosh x} = \frac{\left(\frac{d}{dx}\sinh x\right)(\cosh x) - (\sinh x)\left(\frac{d}{dx}\cosh x\right)}{\cosh^2 x}$$

$$= \frac{(\cosh x)(\cosh x) - (\sinh x)(\sinh x)}{\cosh^2 x} = \frac{\left(\frac{e^x + e^{-x}}{2}\right)^2 - \left(\frac{e^x - e^{-x}}{2}\right)^2}{\cosh^2 x} = \frac{1}{\cosh^2 x} = \operatorname{sech}^2 x$$

$$\frac{d}{dx}\coth x = \frac{d}{dx}\frac{1}{\tanh x} = -\frac{1}{\tanh^2 x}\left(\frac{d}{dx}\tanh x\right) = -\frac{1}{\tanh^2 x}(\operatorname{sech}^2 x) = -\operatorname{csch}^2 x$$

$$\frac{d}{dx}\operatorname{sech} x = \frac{d}{dx}\frac{1}{\cosh x} = -\frac{1}{\cosh^2 x}\left(\frac{d}{dx}\cosh x\right) = -\frac{1}{\cosh^2 x}(\sinh x) = -\operatorname{sech} x \tanh x$$

$$\frac{d}{dx}\operatorname{csch} x = \frac{d}{dx}\frac{1}{\sinh x} = -\frac{1}{\sinh^2 x}\left(\frac{d}{dx}\sinh x\right) = -\frac{1}{\sinh^2 x}(\cosh x) = -\operatorname{csch} x \coth x$$

6. (a) If $PQ = 1013$, then $Q = \frac{1013}{P}$, so

$$\frac{dQ}{dP} = -\frac{1013}{P^2}.$$

Therefore

$$E_d = \frac{P}{Q}\frac{dQ}{dP} = \frac{P}{\left(\frac{1013}{P}\right)}\left(-\frac{1013}{P^2}\right) = -1.$$

<u>*Alternative solution*</u>: Differentiating both sides of the equation $PQ = 1013$ with respect to P, we have

$$Q + P\frac{dQ}{dP} = 0.$$

This implies that $P\frac{dQ}{dP} = -Q$, and so

$$E_d = \frac{P}{Q}\frac{dQ}{dP} = -\frac{Q}{Q} = -1.$$

(b) If $Q = 1013e^{-P}$, then $\frac{dQ}{dP} = 1013(-e^{-P}) = -1013e^{-P}$. Thus

$$E_d = \frac{P}{Q}\frac{dQ}{dP} = \frac{P}{1013e^{-P}}(-1013e^{-P}) = -P.$$

(c) If $Q = 1013 - P^a$, then $\frac{dQ}{dP} = -aP^{a-1}$. Thus

$$E_d = \frac{P}{Q}\frac{dQ}{dP} = \frac{P}{1013 - P^a}(-aP^{a-1}) = \frac{-aP^a}{1013 - P^a} = \frac{a}{1 - 1013P^{-a}}.$$

7. Since f is continuous at 0, we have $\lim_{x\to 0} f(x) = f(0)$. Now

$$f(0) = be^0 - 0 = b$$
$$\lim_{x\to 0^+} f(x) = \lim_{x\to 0^+}(be^x - x) = be^0 - 0 = b$$

So we must also require that $\lim_{x\to 0^-} f(x) = b$.

We claim that this requirement implies that $a = 1$. This is because if $a \neq 1$, then

$$\lim_{x\to 0^-} f(x) = \lim_{x\to 0^-} \frac{\sin 2x + \sin x}{e^x - a} = \frac{\sin(2\cdot 0) + \sin 0}{e^0 - a} = \frac{0}{1-a} = 0 \neq b.$$

Therefore indeed $a = 1$.

Now since $a = 1$, we have

$$\lim_{x\to 0^-} f(x) = \lim_{x\to 0^-} \frac{\sin 2x + \sin x}{e^x - 1} = \lim_{x\to 0^-}\left(\frac{\sin 2x + \sin x}{x}\cdot\frac{x}{e^x - 1}\right)$$

$$= \lim_{x\to 0^-}\left[\left(2\cdot\frac{\sin 2x}{2x} + \frac{\sin x}{x}\right)\cdot\frac{x}{e^x - 1}\right]$$

$$= (2\cdot 1 + 1)\cdot 1 = 3.$$

Therefore $b = 3$.

8. Since the right-hand derivative

$$f'_{-}(0) = \lim_{x\to 0^-} \frac{f(x) - f(0)}{x} = \lim_{x\to 0^-} \frac{\sin 2x \cos 2x - 1}{x} = +\infty$$

is not a finite real number, it follows that $f'(0)$ does not exist.

<u>*Alternative solution*</u>: Since

$$\lim_{x\to 0^+} f(x) = \lim_{x\to 0^+} e^{2x} = 1$$

but

$$\lim_{x\to 0^-} f(x) = \lim_{x\to 0^-} \sin 2x \cos 2x = 0,$$

it follows that f is not continuous at 0 and so $f'(0)$ cannot exist.

9. (i)

$$\frac{d}{dx}\left[f\left(\sqrt[3]{g(x)}\right)f(x)\right]^2\Bigg|_{x=0}$$

$$= 2\left[f\left(\sqrt[3]{g(x)}\right)f(x)\right]\cdot\left[f'\left(\sqrt[3]{g(x)}\right)\frac{1}{3[g(x)]^{\frac{2}{3}}}g'(x)\cdot f(x) + f\left(\sqrt[3]{g(x)}\right)\cdot f'(x)\right]\Bigg|_{x=0}$$

$$= 2\left[f\left(\sqrt[3]{g(0)}\right)f(0)\right]\cdot\left[f'\left(\sqrt[3]{g(0)}\right)\frac{1}{3[g(0)]^{\frac{2}{3}}}g'(0)\cdot f(0) + f\left(\sqrt[3]{g(0)}\right)\cdot f'(0)\right]$$

$$= 2\left[f\left(\sqrt[3]{1}\right)\cdot 2\right]\cdot\left[f'\left(\sqrt[3]{1}\right)\frac{1}{3(1)^{\frac{2}{3}}}(0)\cdot 2 + f\left(\sqrt[3]{1}\right)\cdot(-4)\right]$$

$$= 2[f(1)\cdot 2]\cdot[f(1)\cdot(-4)]$$

$$= 2(3\cdot 2)\cdot[3\cdot(-4)] = -144$$

(ii)

$$\frac{d}{dx}\frac{g([f(x)]^2+1)}{f\big(f(x)\big)}\Bigg|_{x=-1}$$

$$= \frac{g'([f(x)]^2+1)[2f(x)f'(x)]\cdot f\big(f(x)\big) - g([f(x)]^2+1)\cdot f'\big(f(x)\big)f'(x)}{\left[f\big(f(x)\big)\right]^2}\Bigg|_{x=-1}$$

$$= \frac{g'([f(-1)]^2+1)[2f(-1)f'(-1)]\cdot f\big(f(-1)\big) - g([f(-1)]^2+1)\cdot f'\big(f(-1)\big)f'(-1)}{\left[f\big(f(-1)\big)\right]^2}$$

$$= \frac{g'(1^2+1)[2(1)(2)]\cdot f(1) - g(1^2+1)\cdot f'(1)f'(-1)}{[f(1)]^2}$$

$$= \frac{g'(2)[2(1)(2)]\cdot f(1) - g(2)\cdot f'(1)f'(-1)}{[f(1)]^2}$$

$$= \frac{(-5)[2(1)(2)]\cdot 3 - 4\cdot(-3)(2)}{3^2} = -4$$

10. Differentiating both sides of the equation of the curve, we have $\frac{1}{2\sqrt{x}} + \frac{1}{2\sqrt{y}}\frac{dy}{dx} = 0$, so

$$\frac{dy}{dx} = -\frac{\sqrt{y}}{\sqrt{x}}.$$

Now except for the two end points $(0, a)$ and $(a, 0)$, each point on the curve has coordinates $\left(t, \left(\sqrt{a} - \sqrt{t}\right)^2\right)$ for some $t \in (0, a)$. The slope of the tangent line to the curve at this point is given by

$$\left.\frac{dy}{dx}\right|_{\left(t,\left(\sqrt{a}-\sqrt{t}\right)^2\right)} = -\frac{\sqrt{a} - \sqrt{t}}{\sqrt{t}} = 1 - \frac{\sqrt{a}}{\sqrt{t}}.$$

The equation of the tangent line to the curve at this point is therefore

$$y - \left(\sqrt{a} - \sqrt{t}\right)^2 = \left(1 - \frac{\sqrt{a}}{\sqrt{t}}\right)(x - t),$$

i.e.

$$\frac{x}{\sqrt{t}} + \frac{y}{\sqrt{a} - \sqrt{t}} = \sqrt{a}.$$

The y-intercept of the tangent is $\sqrt{a}\left(\sqrt{a} - \sqrt{t}\right)$, while the x-intercept of the tangent is $\sqrt{at}$. Therefore the sum of intercepts is $\sqrt{at} + \sqrt{a}\left(\sqrt{a} - \sqrt{t}\right) = a$. ∎

11. (a) Differentiating both sides with respect to x, we have

$$(1)(y^2) + (x)\left(2y\frac{dy}{dx}\right) + e^y\frac{dy}{dx} = [-\sin(x + y^2)]\left(1 + 2y\frac{dy}{dx}\right)$$

$$[2xy + e^y + 2y\sin(x + y^2)]\frac{dy}{dx} = -y^2 - \sin(x + y^2)$$

$$\frac{dy}{dx} = -\frac{y^2 + \sin(x + y^2)}{2xy + e^y + 2y\sin(x + y^2)}.$$

(b) Differentiating both sides with respect to x, we have

$$2x + 2y\frac{dy}{dx} - \frac{1}{\sqrt{1 - y^2}}\frac{dy}{dx} = 0$$

$$\frac{dy}{dx} = \frac{2x}{\frac{1}{\sqrt{1 - y^2}} - 2y} = \frac{2x\sqrt{1 - y^2}}{1 - 2y\sqrt{1 - y^2}}.$$

Alternative solution: The given equation can be rewritten as $y = \sin(x^2 + y^2)$. Differentiating both sides with respect to x, we have

$$\frac{dy}{dx} = \cos(x^2 + y^2)\left(2x + 2y\frac{dy}{dx}\right), \qquad \text{i.e.} \qquad \frac{dy}{dx} = \frac{2x\cos(x^2 + y^2)}{1 - 2y\cos(x^2 + y^2)}.$$

(c) Differentiating both sides with respect to x, we have

$$\frac{1}{2}\frac{1}{x^2+y^2}\left(2x+2y\frac{dy}{dx}\right)+\frac{1}{1+\left(\frac{y}{x}\right)^2}\left(\frac{x\frac{dy}{dx}-y}{x^2}\right)=0$$

$$\frac{1}{x^2+y^2}\left(x+y\frac{dy}{dx}+x\frac{dy}{dx}-y\right)=0$$

$$\frac{dy}{dx}=\frac{y-x}{y+x}.$$

12. (a) For every $x\in\mathbb{R}$, we have

$$\begin{aligned} f'(x) &= (2^x\ln 2)(x\sin x+\cos x)+(2^x)[(\sin x+x\cos x)+(-\sin x)] \\ &= 2^x[(\ln 2)x\sin x+(\ln 2)\cos x+x\cos x]. \end{aligned}$$

(ii) For every $x\in\mathbb{R}$, we have both $1+x+x^2>0$ and $1-x+2x^2>0$ (why?). Taking natural logarithm on both sides, we have

$$\ln f(x)=\frac{1}{2}\ln\left(\frac{1+x+x^2}{1-x+2x^2}\right)=\frac{1}{2}\ln(1+x+x^2)-\frac{1}{2}\ln(1-x+2x^2).$$

Now differentiating both sides with respect to x, we get

$$\frac{f'(x)}{f(x)}=\frac{1}{2}\left(\frac{1+2x}{1+x+x^2}\right)-\frac{1}{2}\left(\frac{-1+4x}{1-x+2x^2}\right).$$

Therefore

$$\begin{aligned} f'(x) &= f(x)\left[\frac{1}{2}\left(\frac{1+2x}{1+x+x^2}\right)-\frac{1}{2}\left(\frac{-1+4x}{1-x+2x^2}\right)\right] \\ &= \frac{1}{2}\sqrt{\frac{1+x+x^2}{1-x+2x^2}}\left(\frac{1+2x}{1+x+x^2}+\frac{1-4x}{1-x+2x^2}\right). \end{aligned}$$

(iii) If $x\in(2n\pi,(2n+1)\pi)$ for some non-negative integer n, then we have $x>0$, $\sin x>0$ and $e^{-x}<1$. Taking natural logarithm on both sides, we have

$$\ln f(x)=\frac{1}{2}\ln\left[(x\sin x)\sqrt{1-e^{-x}}\right]=\frac{1}{2}\ln x+\frac{1}{2}\ln\sin x+\frac{1}{4}\ln(1-e^{-x}).$$

Differentiating both sides with respect to x, we get

$$\frac{f'(x)}{f(x)}=\frac{1}{2x}+\frac{1}{2\sin x}\cos x+\frac{1}{4(1-e^{-x})}(e^{-x})=\frac{1}{2x}+\frac{1}{2}\cot x+\frac{e^{-x}}{4(1-e^{-x})}.$$

Therefore

$$\begin{aligned} f'(x) &= f(x)\left[\frac{1}{2x}+\frac{1}{2}\cot x+\frac{e^{-x}}{4(1-e^{-x})}\right] \\ &= \sqrt{(x\sin x)\sqrt{1-e^{-x}}}\left[\frac{1}{2x}+\frac{1}{2}\cot x+\frac{e^{-x}}{4(1-e^{-x})}\right]. \end{aligned}$$

(iv) For every $x > 1$, we have $\ln x > 0$. Taking natural logarithm on both sides, we have

$$\ln f(x) = \frac{1}{\ln x}\ln\ln x = \frac{\ln\ln x}{\ln x}.$$

Differentiating both sides with respect to x, we get

$$\frac{f'(x)}{f(x)} = \frac{\left(\frac{1}{\ln x}\frac{1}{x}\right)(\ln x) - (\ln\ln x)\left(\frac{1}{x}\right)}{(\ln x)^2} = \frac{1-\ln\ln x}{x(\ln x)^2}.$$

Therefore

$$f'(x) = f(x)\frac{1-\ln\ln x}{x(\ln x)^2} = (\ln x)^{\frac{1}{\ln x}-2}\frac{1-\ln\ln x}{x}.$$

13. (a) $f(x) = e^{e^e}$ is a constant function, so $f'(x) = 0$ for every $x \in \mathbb{R}$.

(b) According to "power rule", for every $x > 0$ we have

$$f'(x) = e^e x^{e^e - 1}.$$

(c) For every $x > 0$, we have

$$f'(x) = \left(e^{x^e}\right)\left(\frac{d}{dx}x^e\right) = \left(e^{x^e}\right)(ex^{e-1}) = e^{x^e+1}x^{e-1}.$$

(d) For every $x \in \mathbb{R}$, we have

$$f'(x) = \left(e^{e^x}\right)\left(\frac{d}{dx}e^x\right) = \left(e^{e^x}\right)(e^x) = e^{e^x+x}.$$

(v) For every $x > 0$, taking natural logarithm on both sides, we have

$$\ln f(x) = x^e \ln x.$$

Differentiating both sides with respect to x, we get

$$\frac{f'(x)}{f(x)} = (ex^{e-1})(\ln x) + (x^e)\left(\frac{1}{x}\right) = x^{e-1}(1 + e\ln x).$$

Therefore

$$f'(x) = f(x)[x^{e-1}(1 + e\ln x)] = x^{x^e+e-1}(1 + e\ln x).$$

(vi) For every $x > 0$, taking natural logarithm on both sides, we have

$$\ln f(x) = e^x \ln x.$$

Differentiating both sides with respect to x, we get

$$\frac{f'(x)}{f(x)} = (e^x)(\ln x) + (e^x)\left(\frac{1}{x}\right) = e^x\left(\ln x + \frac{1}{x}\right).$$

Therefore

$$f'(x) = f(x)\left[e^x\left(\ln x + \frac{1}{x}\right)\right] = x^{e^x}e^x\left(\ln x + \frac{1}{x}\right).$$

To prepare for (g) and (h) we let $g(x) = x^x$ and first compute $g'(x)$ (cf. Example 3.49). For every $x > 0$, taking natural logarithm on both sides, we have

$$\ln g(x) = x \ln x\,.$$

Differentiating both sides with respect to x, we get

$$\frac{g'(x)}{g(x)} = (1)(\ln x) + (x)\left(\frac{1}{x}\right) = \ln x + 1.$$

Therefore

$$g'(x) = g(x)(\ln x + 1) = x^x(\ln x + 1).$$

(vii) For every $x > 0$, we have

$$f'(x) = \left(e^{x^x}\right)\left(\frac{d}{dx}x^x\right) = e^{x^x}x^x(\ln x + 1).$$

(viii) For every $x > 0$, taking natural logarithm on both sides, we have

$$\ln f(x) = x^x \ln x\,.$$

Differentiating both sides with respect to x, we get

$$\frac{f'(x)}{f(x)} = \left(\frac{d}{dx}x^x\right)(\ln x) + (x^x)\left(\frac{1}{x}\right) = \left(x^x(\ln x + 1)\right)(\ln x) + (x^x)\left(\frac{1}{x}\right) = x^x\left[(\ln x)^2 + \ln x + \frac{1}{x}\right].$$

Therefore

$$f'(x) = f(x)x^x\left[(\ln x)^2 + \ln x + \frac{1}{x}\right] = x^{x^x + x}\left[(\ln x)^2 + \ln x + \frac{1}{x}\right].$$

14. (a) For every $x \in \mathbb{R} \setminus \left\{\left(n + \frac{1}{2}\right)\pi : n \in \mathbb{Z}\right\}$, we have $\sin x \in (-1, 1)$. So

$$\begin{aligned} f'(x) &= \frac{1}{2} \cdot \frac{1}{\frac{1 + \sin x}{1 - \sin x}} \cdot \frac{(\cos x)(1 - \sin x) - (1 + \sin x)(-\cos x)}{(1 - \sin x)^2} \\ &= \frac{2\cos x}{2(1 + \sin x)(1 - \sin x)} = \frac{\cos x}{1 - \sin^2 x} = \frac{\cos x}{\cos^2 x} = \sec x. \end{aligned}$$

(b) For each of those $x \in \mathbb{R}$ such that $\cos x > 0$, we have

$$f'(x) = \frac{1}{\sec x + \tan x}(\sec x \tan x + \sec^2 x) = (\sec x)\frac{\sec x + \tan x}{\sec x + \tan x} = \sec x.$$

(c) For each of those $x \in \mathbb{R}$ such that $\cos x > 0$, we have

$$\begin{aligned} f'(x) &= \frac{1}{\tan\left(\frac{x}{2} + \frac{\pi}{4}\right)}\left(\frac{1}{2}\sec^2\left(\frac{x}{2} + \frac{\pi}{4}\right)\right) = \frac{1}{2\sin\left(\frac{x}{2} + \frac{\pi}{4}\right)\cos\left(\frac{x}{2} + \frac{\pi}{4}\right)} \\ &= \frac{1}{\sin\left(2\left(\frac{x}{2} + \frac{\pi}{4}\right)\right)} = \frac{1}{\sin\left(x + \frac{\pi}{2}\right)} = \frac{1}{\cos x} = \sec x. \end{aligned}$$

15. (i) For every $x \in \mathbb{R}$, let $y = \arctan x$. Then $y \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ and $x = \tan y$, so

$$\frac{dx}{dy} = \sec^2 y\,.$$

Therefore

$$\frac{dy}{dx} = \frac{1}{\sec^2 y} = \frac{1}{1+\tan^2 y} = \frac{1}{1+x^2}.$$

(ii) For every $x \in (-\infty, -1) \cup (1, +\infty)$, let $y = \operatorname{arcsec} x$. Then $x = \sec y$, so

$$\frac{dx}{dy} = \sec y \tan y\,.$$

We consider the following two cases.

⦿ If $x \in (1, +\infty)$, then $y \in \left(0, \frac{\pi}{2}\right)$. Thus $\tan y > 0$ and so

$$\frac{dy}{dx} = \frac{1}{\sec y \tan y} = \frac{1}{\sec y\sqrt{\sec^2 y - 1}} = \frac{1}{x\sqrt{x^2-1}}.$$

⦿ If $x \in (-\infty, -1)$, then $y \in \left(\frac{\pi}{2}, \pi\right)$. Thus $\tan y < 0$ and so

$$\frac{dy}{dx} = \frac{1}{\sec y \tan y} = \frac{1}{\sec y\left(-\sqrt{\sec^2 y - 1}\right)} = \frac{1}{-x\sqrt{x^2-1}}.$$

Combining the two cases, we have

$$\frac{dy}{dx} = \frac{1}{|x|\sqrt{x^2-1}} \qquad \text{for every } x \in (-\infty, -1) \cup (1, +\infty).$$

16. (i)

$$\left.\frac{d}{dx}\left[(f^{-1})'(x)(g')^{-1}(x)\right]\right|_{x=0}$$

$$= \left.\frac{d}{dx}\frac{(g')^{-1}(x)}{f'\big(f^{-1}(x)\big)}\right|_{x=0} = \left.\frac{\left[\frac{d}{dx}(g')^{-1}(x)\right]f'\big(f^{-1}(x)\big) - (g')^{-1}(x)\left[\frac{d}{dx}f'\big(f^{-1}(x)\big)\right]}{\left[f'\big(f^{-1}(x)\big)\right]^2}\right|_{x=0}$$

$$= \left.\frac{\left[\frac{1}{g''\big((g')^{-1}(x)\big)}\right]f'\big(f^{-1}(x)\big) - (g')^{-1}(x)\left[f''\big(f^{-1}(x)\big)\cdot\frac{1}{f'\big(f^{-1}(x)\big)}\right]}{\left[f'\big(f^{-1}(x)\big)\right]^2}\right|_{x=0}$$

$$= \frac{\frac{f'\big(f^{-1}(0)\big)}{g''\big((g')^{-1}(0)\big)} - \frac{(g')^{-1}(0)f''\big(f^{-1}(0)\big)}{f'\big(f^{-1}(0)\big)}}{\left[f'\big(f^{-1}(0)\big)\right]^2} = \frac{\frac{f'(1)}{g''(-2)} - \frac{(-2)f''(1)}{f'(1)}}{[f'(1)]^2} = \frac{\frac{3}{-1} - \frac{(-2)(9)}{3}}{3^2} = \frac{1}{3}$$

(ii)

$$\frac{d}{dx}(g\circ f')^{-1}\big(f^{-1}(x)\big)\Big|_{x=-2}$$

$$=\frac{d}{dx}\big((f')^{-1}\circ g^{-1}\big)\big(f^{-1}(x)\big)\Big|_{x=-2}=\big((f')^{-1}\big)'\left(g^{-1}\big(f^{-1}(x)\big)\right)\cdot(g^{-1})'\big(f^{-1}(x)\big)\cdot(f^{-1})'(x)\Big|_{x=-2}$$

$$=\frac{1}{f''\left((f')^{-1}\left(g^{-1}\big(f^{-1}(-2)\big)\right)\right)}\cdot\frac{1}{g'\left(g^{-1}\big(f^{-1}(-2)\big)\right)}\cdot\frac{1}{f'\big(f^{-1}(-2)\big)}$$

$$=\frac{1}{f''\left((f')^{-1}\big(g^{-1}(0)\big)\right)}\cdot\frac{1}{g'\big(g^{-1}(0)\big)}\cdot\frac{1}{f'(0)}=\frac{1}{f''\big((f')^{-1}(3)\big)}\cdot\frac{1}{g'(3)}\cdot\frac{1}{f'(0)}$$

$$=\frac{1}{f''(1)}\cdot\frac{1}{g'(3)}\cdot\frac{1}{f'(0)}=\frac{1}{9}\cdot\frac{1}{-2}\cdot\frac{1}{4}=-\frac{1}{72}$$

17. (a) We observe that

$$f'(x)=3e^{3x},\qquad f''(x)=3^2e^{3x},\qquad f'''(x)=3^3e^{3x}$$

and so on. Inductively, we have

$$f^{(n)}(x)=3^ne^{3x}.$$

(b) We observe that

$$g'(x)=x^{-1},\qquad g''(x)=(-1)x^{-2},\qquad g'''(x)=(-1)(-2)x^{-3},\qquad g^{(4)}(x)=(-1)(-2)(-3)x^{-4}$$

and so on. Inductively, we have

$$g^{(n)}(x)=(-1)(-2)\cdots(-n+1)x^{-n}.$$

Using factorial notation, we can write

$$g^{(n)}(x)=\frac{(-1)^{n-1}(n-1)!}{x^n}.$$

(c) We observe that

$$h'(x)=a\cos ax\,,\qquad h''(x)=-a^2\sin ax\,,\qquad h'''(x)=-a^3\cos ax\,,\qquad h^{(4)}(x)=a^4\sin ax$$

and so on. Inductively, we have

$$h^{(n)}(x)=\begin{cases}a^n\cos ax & \text{if } n=4m+1 \text{ for some non-negative integer } m\\ -a^n\sin ax & \text{if } n=4m+2 \text{ for some non-negative integer } m\\ -a^n\cos ax & \text{if } n=4m+3 \text{ for some non-negative integer } m\\ a^n\sin ax & \text{if } n=4m \text{ for some non-negative integer } m\end{cases}$$

$$=a^n\sin\left(ax+\frac{n\pi}{2}\right).$$

18. Differentiating both sides with respect to x, we have $\frac{1}{2\sqrt{y}}\frac{dy}{dx}+y+x\frac{dy}{dx}=0$, so

$$\frac{dy}{dx}=\frac{-y}{x+\frac{1}{2\sqrt{y}}}=\frac{-2y^{\frac{3}{2}}}{2x\sqrt{y}+1}.$$

Differentiating once more with respect to x, we get

$$\begin{aligned}\frac{d^2y}{dx^2}&=\frac{\left(-3\sqrt{y}\frac{dy}{dx}\right)\left(2x\sqrt{y}+1\right)-\left(-2y^{\frac{3}{2}}\right)\left(2\sqrt{y}+2x\cdot\frac{1}{2\sqrt{y}}\frac{dy}{dx}\right)}{\left(2x\sqrt{y}+1\right)^2}\\&=\frac{\left(-3\sqrt{y}\frac{dy}{dx}\right)\left(2x\sqrt{y}+1\right)+4y^2+2xy\frac{dy}{dx}}{\left(2x\sqrt{y}+1\right)^2}\\&=\frac{\left(-3\sqrt{y}\right)\left(-2y^{\frac{3}{2}}\right)+4y^2+2xy\left(\frac{-2y^{\frac{3}{2}}}{2x\sqrt{y}+1}\right)}{\left(2x\sqrt{y}+1\right)^2}\\&=\frac{2y^2\left(8x\sqrt{y}+5\right)}{\left(2x\sqrt{y}+1\right)^3}.\end{aligned}$$

19. By chain rule, the net force acting on the object is given by

$$F=\frac{d}{dt}(mv)=\frac{d}{dt}\frac{m_0v}{\sqrt{1-\frac{v^2}{c^2}}}=\left(\frac{d}{dv}\frac{m_0v}{\sqrt{1-\frac{v^2}{c^2}}}\right)\frac{dv}{dt}.$$

Now by quotient rule, we have

$$\begin{aligned}\frac{d}{dv}\frac{m_0v}{\sqrt{1-\frac{v^2}{c^2}}}&=\frac{(m_0)\sqrt{1-\frac{v^2}{c^2}}-(m_0v)\left(\frac{1}{2\sqrt{1-v^2/c^2}}\cdot\frac{2v}{c^2}\right)}{\left(\sqrt{1-\frac{v^2}{c^2}}\right)^2}\\&=\frac{(m_0)\left(1-\frac{v^2}{c^2}\right)-m_0\left(\frac{v^2}{c^2}\right)}{\left(1-\frac{v^2}{c^2}\right)^{\frac{3}{2}}}=\frac{m_0}{\left(1-\frac{v^2}{c^2}\right)^{\frac{3}{2}}}.\end{aligned}$$

Therefore

$$F=\frac{m_0}{\left(1-\frac{v^2}{c^2}\right)^{\frac{3}{2}}}\frac{dv}{dt}=\frac{m_0a}{\left(1-\frac{v^2}{c^2}\right)^{\frac{3}{2}}}.$$

■

20. (a) Given the position $x(t) = \sqrt{3}\sin t + \cos t$ of the object, its velocity and its acceleration after t seconds are respectively given by

$$v(t) = x'(t) = \sqrt{3}\cos t - \sin t \qquad \text{and} \qquad a(t) = v'(t) = -\sqrt{3}\sin t - \cos t.$$

When $t = \frac{\pi}{6}$, we have

$$v\left(\frac{\pi}{6}\right) = \sqrt{3}\cos\frac{\pi}{6} - \sin\frac{\pi}{6} = 1 \qquad \text{and} \qquad a\left(\frac{\pi}{6}\right) = -\sqrt{3}\sin\frac{\pi}{6} - \cos\frac{\pi}{6} = -\sqrt{3},$$

so the velocity and the acceleration of the object at $t = \frac{\pi}{6}$ are 1 unit per second and $-\sqrt{3}$ units per second squared respectively.

(b) Since $x(t) = a\sin t + b\cos t$, we have

$$x'(t) = a\cos t - b\sin t$$

for every $t \in \mathbb{R}$. Therefore

$$\begin{aligned}
[x(t)]^2 + [x'(t)]^2 &= (a\sin t + b\cos t)^2 + (a\cos t - b\sin t)^2 \\
&= a^2\sin^2 t + 2ab\sin t\cos t + b^2\cos^2 t + a^2\cos^2 t - 2ab\sin t\cos t + b^2\sin^2 t \\
&= a^2(\sin^2 t + \cos^2 t) + b^2(\sin^2 t + \cos^2 t) \\
&= a^2 + b^2
\end{aligned}$$

for every $t \in \mathbb{R}$, which is always constant. ■

21. (a) (i) Given $V = 1000e^{-10r}$, we have

$$\frac{dV}{dr} = 1000(-10e^{-10r}) \qquad \text{and} \qquad \frac{d^2V}{dr^2} = 1000(100e^{-10r}).$$

Thus the duration and the convexity are respectively given by

$$D = -\frac{1}{V}\frac{dV}{dr} = 10 \qquad \text{and} \qquad C = \frac{1}{V}\frac{d^2V}{dr^2} = 100,$$

which are constant functions of r. When $r = 0.06$, we still have $D = 10$ and $C = 100$.

(ii) Given $V = \frac{1000}{(1+r)^{10}}$, we have

$$\frac{dV}{dr} = 1000 \cdot \frac{-10}{(1+r)^{11}} \qquad \text{and} \qquad \frac{d^2V}{dr^2} = 1000 \cdot \frac{10 \cdot 11}{(1+r)^{12}}.$$

Thus the duration and the convexity are respectively given by

$$D = -\frac{1}{V}\frac{dV}{dr} = \frac{10}{1+r} \qquad \text{and} \qquad C = \frac{1}{V}\frac{d^2V}{dr^2} = \frac{110}{(1+r)^2}.$$

When $r = 0.06$, we have $D \approx 9.43$ and $C \approx 97.90$.

(iii) Given $V = \frac{250}{(1+r)^5} + \frac{1250}{(1+r)^{10}}$, we have

$$\frac{dV}{dr} = \frac{-5 \cdot 250}{(1+r)^6} + \frac{-10 \cdot 1250}{(1+r)^{11}} \qquad \text{and} \qquad \frac{d^2V}{dr^2} = \frac{5 \cdot 6 \cdot 250}{(1+r)^7} + \frac{10 \cdot 11 \cdot 1250}{(1+r)^{12}}.$$

Thus the duration and the convexity are respectively given by

$$D = -\frac{1}{V}\frac{dV}{dr} = \frac{1}{1+r}\frac{5(1+r)^5 + 50}{(1+r)^5 + 5} \qquad \text{and} \qquad C = \frac{1}{V}\frac{d^2V}{dr^2} = \frac{1}{(1+r)^2}\frac{30(1+r)^5 + 550}{(1+r)^5 + 5}.$$

When $r = 0.06$, we have $D \approx 8.44$ and $C \approx 82.87$.

(b) By product rule and chain rule, we have

$$\begin{aligned}\frac{dD}{dr} &= \frac{d}{dr}\left(-\frac{1}{V}\frac{dV}{dr}\right) \\ &= \left(\frac{1}{V^2}\frac{dV}{dr}\right)\left(\frac{dV}{dr}\right) + \left(-\frac{1}{V}\right)\left(\frac{d^2V}{dr^2}\right) \\ &= \frac{1}{V^2}\left(\frac{dV}{dr}\right)^2 - \frac{1}{V}\frac{d^2V}{dr^2} \\ &= \left(-\frac{1}{V}\frac{dV}{dr}\right)^2 - \frac{1}{V}\frac{d^2V}{dr^2} \\ &= D^2 - C.\end{aligned}$$

Therefore $C = D^2 - \frac{dD}{dr}$. ■

22. Let r, A and V be the radius, surface area and volume of the spherical capsule respectively. We have

$$V = \frac{4}{3}\pi r^3 \qquad \text{and} \qquad A = 4\pi r^2.$$

Differentiating both sides of both equalities with respect to time t, we get

$$\frac{dV}{dt} = 4\pi r^2 \frac{dr}{dt} \qquad \text{and} \qquad \frac{dA}{dt} = 8\pi r \frac{dr}{dt}.$$

When the radius is 4 cm, it is given that $\left.\frac{dV}{dt}\right|_{r=4} = -9\pi$; so we have

$$-9\pi = 4\pi(4^2)\left.\frac{dr}{dt}\right|_{r=4} \qquad \text{and} \qquad \left.\frac{dA}{dt}\right|_{r=4} = 8\pi(4)\left.\frac{dr}{dt}\right|_{r=4}.$$

Therefore $\left.\frac{dr}{dt}\right|_{r=4} = -\frac{9}{64}$ and $\left.\frac{dA}{dt}\right|_{r=4} = -\frac{9}{2}\pi$.

So when the radius is 4 cm, the radius is decreasing at a rate of $\frac{9}{64}$ $\mathrm{cm\ s^{-1}}$ and the surface area is decreasing at a rate of $\frac{9}{2}\pi$ $\mathrm{cm^2\ s^{-1}}$.

23. Let the length of OA be l units. Note that when θ is given, the coordinates of A are given by (h,k), where h and k satisfy

$$k = h^2 \qquad \text{and} \qquad k = h\tan\theta.$$

These equations imply $h = \tan\theta$ and $k = \tan^2\theta$. Pythagoras' Theorem then gives the relation

$$\begin{aligned} l^2 = h^2 + k^2 &= (\tan\theta)^2 + (\tan^2\theta)^2 \\ &= \tan^2\theta + \tan^4\theta \\ &= \tan^2\theta\sec^2\theta. \end{aligned}$$

(a) Now let the area of ΔOBA be S square units. Our first aim is to relate S with θ. We have

$$S = \frac{1}{2}(l\cos\theta)(l\sin\theta) = \frac{1}{2}l^2\sin\theta\cos\theta$$

$$= \frac{1}{2}(\tan^2\theta\sec^2\theta)\sin\theta\cos\theta = \frac{1}{2}\tan^3\theta.$$

Differentiating both sides with respect to time t, we get

$$\frac{dS}{dt} = \frac{3}{2}\tan^2\theta\sec^2\theta\frac{d\theta}{dt}$$

When $\theta = \frac{\pi}{3}$, we have

$$\left.\frac{dS}{dt}\right|_{\theta=\frac{\pi}{3}} = \frac{3}{2}\tan^2\frac{\pi}{3}\sec^2\frac{\pi}{3}\left(\left.\frac{d\theta}{dt}\right|_{\theta=\frac{\pi}{3}}\right) = \frac{3}{2}\left(\sqrt{3}\right)^2(2)^2\left(\frac{1}{3}\right) = 6$$

Therefore the area of ΔOBA is increasing at a rate of 6 square units per minute.

(b) We have the relation $l^2 = \tan^2\theta + \tan^4\theta$. Differentiating both sides with respect to time t, we get

$$2l\frac{dl}{dt} = 2\tan\theta\sec^2\theta\frac{d\theta}{dt} + 4\tan^3\theta\sec^2\theta\frac{d\theta}{dt}$$

$$\frac{dl}{dt} = \frac{1}{l}(1 + 2\tan^2\theta)(\tan\theta)(1 + \tan^2\theta)\frac{d\theta}{dt}.$$

When $l = \sqrt{2}$, we have

$$\begin{aligned} \left(\sqrt{2}\right)^2 &= \tan^2\theta + \tan^4\theta \\ \tan^4\theta + \tan^2\theta - 2 &= 0 \\ (\tan^2\theta + 2)(\tan^2\theta - 1) &= 0 \\ (\tan^2\theta + 2)(\tan\theta + 1)(\tan\theta - 1) &= 0. \end{aligned}$$

Since $\theta \in \left(0, \frac{\pi}{2}\right)$, we must have $\tan\theta > 0$. So the above equation implies that $\tan\theta = 1$. Thus,

$$\left.\frac{dl}{dt}\right|_{l=\sqrt{2}} = \frac{1}{\sqrt{2}}[1 + 2(1)](1)(1 + 1)\left(\frac{1}{3}\right) = \sqrt{2}.$$

Therefore the length of OA is increasing at a rate of $\sqrt{2}$ units per minute.